A Class of Strongly Homotopy Lie Algebras with Simplified sh-Lie Structures

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Abstract

It is known that a single mapping defined on one term of a differential graded vector space extends to a strongly homotopy Lie algebra structure on the graded space when that mapping satisfies two conditions. This strongly homotopy Lie algebra is nontrivial (it is not a Lie algebra); however we show that one can obtain an sh-Lie algebra where the only nonzero mappings defining it are the lower order mappings. This structure applies to a significant class of examples. Moreover in this case the graded space can be replaced by another graded space, with only three nonzero terms, on which the same sh-Lie structure exists.

1 Introduction and sh-Lie algebras

Strongly homotopy Lie algebras (sh-Lie algebras/structures) have recently been the focus of study in mathematics [2, 4, 5]. Their applications have appeared in mathematics [6], in mathematical physics [2, 3], and in physics [7]. We first present some background material. Then in section 2 we prove the main result. Finally in section 3, we present three examples, two of which have appeared in two different applications.

To begin with our discussion, let X_* be a graded vector space with a differential (lowering the degree by 1) l_1, and maps $\eta : X_n \rightarrow H_n, \lambda : H_n \rightarrow X_n$, and $s : X_n \rightarrow X_{n+1}$ (i.e. s raises the degree by 1), where H_* is the

homology complex of the complex X_*, H_0 is generally non-trivial and $H_n = 0$ for $n > 0$. We also assume that

$$\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1. \tag{1.1}$$

Notice that 1.1 simplifies to $\lambda \circ \eta - 1_{X_*} = l_1 \circ s$ in degree 0, and $-1_{X_*} = l_1 \circ s + s \circ l_1$ in degree > 0. For more details we refer the reader to [2]. The following is the formal definition of sh-Lie structures (see [2])

Definition An sh-Lie structure on a graded vector space X_* is a collection of linear, skew-symmetric maps $l_k : \bigotimes^k X_* \to X_*$ of degree $k-2$ that satisfy the relation

$$\sum_{i+j=n+1} \sum_{unsh(i,n-i)} e(\sigma)(-1)^{\sigma}(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), \cdots, x_{\sigma(n)}) = 0,$$

where $1 \leq i, j$.

Notice that in this definition $e(\sigma)$ is the Koszul sign which depends on the permutation σ as well as on the degree of the elements $x_1, x_2, \cdots, x_n$ (where a minus sign is introduced whenever two consecutive odd elements are permuted, see [4]). Also observe that it is convenient in this context to suppress some of the notation and assume the summands are over the appropriate unshuffles with their corresponding signs (e.g. if $n = 3$ one writes $l_1 l_3 + l_2 l_2 + l_3 l_1 = 0$).

We assume the existence of a linear skew-symmetric map $\tilde{l}_2 : X_0 \bigotimes X_0 \to X_0$ satisfying conditions (i) and (ii) below so that an sh-Lie structure exists, where we quote the following from [2]

Theorem 1.1 *A skew-symmetric linear map $\tilde{l}_2 : X_0 \bigotimes X_0 \to X_0$ that satisfies conditions (i) and (ii) below extends to an sh-Lie structure on the graded space X_*;*
(i) $\quad \tilde{l}_2(c, b_1) = b_2$
(ii) $\quad \displaystyle\sum_{\sigma \in unsh(2,1)} (-1)^{\sigma} \tilde{l}_2(\tilde{l}_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)}) = b_3$
where c, c_1, c_2, c_3 are cycles and b_1, b_2, b_3 are boundaries in X_0.

2 The main theorem

While Theorem 1.1 guarantees the existence of an sh-Lie structure on the graded vector space X_*, we *show* that one can always choose an sh-Lie structure such that
1. $l_2 = 0$ in degree > 1.
2. $l_3 = 0$ in degree > 0.
3. $l_n \equiv 0, n > 3$.

Remark Markl has observed (see [2]) that such an sh-Lie structure exists in the case that $l_2(c, b) = 0$ for each cycle c and boundary b. To our knowledge even the proof of this special case has not been published although the result is known to specialists in the area.

We find it convenient in what follows to refer to the image of a combination of maps by the combination itself: for example l_2l_1 would stand for the image of the map l_2l_1 acting on some element in the appropriate space as the context implies (e.g l_2l_1 may stand for $l_2l_1(x_p \otimes x_q) = l_2(l_1x_p \otimes x_q + (-1)^p x_p \otimes l_1x_q) = l_2(l_1x_p \otimes x_q) + (-1)^p l_2(x_p \otimes l_1x_q)$ for $x_p \in X_p$ and $x_q \in X_q$). l_2l_2 would stand for the three unshuffles of the composite l_2l_2, again acting on an element in the approprite space (where we skip writing the element it's acting on, e.g. $l_2l_2 = l_2l_2(x_0 \otimes x_0' \otimes x_0'') = l_2(l_2(x_0, x_0'), x_0'') - l_2(l_2(x_0, x_0''), x_0') + l_2(l_2(x_0', x_0''), x_0)$ for $x_0 \otimes x_0' \otimes x_0'' \in X_0 \bigotimes X_0 \bigotimes X_0$) ...etc. Let's also quote the following from [2] as it is needed in our proof:

Lemma 2.1 *(i) l_2l_1 is a boundary.*
(ii) $l_2l_2 + l_3l_1$ is a boundary.
(iii) More generally $(\sum_{i+j=n+1, j>1} (-1)^{i(j-1)} l_j l_i)$ *is a boundary.*

To begin with the proof we define l_2 inductively by

$$l_2 = -s \circ l_2l_1$$

(this is just $-sl_2l_1$, i.e. s acts on the image of l_2l_1) where we begin with $l_2 = \tilde{l}_2$ in degree 0, and recall that s satisfies

$$\lambda \circ \eta - 1_{X_*} = l_1 \circ s$$

in degree 0, and

$$-1_{X_*} = l_1 \circ s + s \circ l_1$$

in degree > 0 (see 1.1). One checks that if $l_2 l_1$ is in degree 0 then

$$l_1(-sl_2l_1) = l_2l_1 - \lambda \circ \eta(l_2l_1) = l_2l_1,$$

where $(\lambda \circ \eta)(l_2l_1) = 0$ since l_2l_1 is a boundary. While if l_2l_1 is in degree > 0 then

$$l_1(-sl_2l_1) = l_2l_1 + sl_1(l_2l_1) = l_2l_1,$$

where $l_1(l_2l_1) = 0$ since l_2l_1 is a boundary. So we have a well-defined chain map l_2 satisfying $l_1l_2 = l_2l_1$. Now we show that l_2 as defined above is *zero* in degree > 1.

First consider l_2 on $\underline{X_1 \otimes X_1}$: take an element $x_1 \otimes x_1' \in X_1 \otimes X_1$. We have $l_2(x_1 \otimes x_1') = -s\{l_2(l_1x_1 \otimes x_1' - x_1 \otimes l_1x_1')\}$. But $l_2(l_1x_1 \otimes x_1') = l_2(x_1 \otimes l_1x_1')$, since by definition

$$l_2(l_1x_1 \otimes x_1') = -sl_2l_1(l_1x_1 \otimes x_1') = -sl_2(l_1x_1 \otimes l_1x_1'),$$

and

$$l_2(x_1 \otimes l_1x_1') = -sl_2l_1(x_1 \otimes l_1x_1') = -sl_2(l_1x_1 \otimes l_1x_1').$$

So $l_2 = 0$ on $X_1 \otimes X_1$. Now consider l_2 on $\underline{X_2 \otimes X_0}$: take $x_2 \otimes x_0 \in X_2 \otimes X_0$. Then $l_2(x_2 \otimes x_0) = -sl_2(l_1x_2 \otimes x_0)$, but $l_2(l_1x_2 \otimes x_0) = -sl_2l_1(l_1x_2 \otimes x_0) = -sl_2(0) = 0$. So $l_2 = 0$ on $X_2 \otimes X_0$.

Proceeding by induction one then shows that $l_2 = 0$ on $X_n \otimes X_0$ with $n \geq 3$: $l_2(x_n \otimes x_0) = -sl_2l_1(x_n \otimes x_0) = -sl_2(l_1x_n \otimes x_0) = -s(0) = 0$. On the other hand consider l_2 on $X_n \otimes X_m, n > 1, m \geq 1$: $l_2(x_n \otimes x_m) = -sl_2l_1(x_n \otimes x_m) = -sl_2(l_1x_n \otimes x_m + (-1)^n x_n \otimes l_1x_m) = -s(0) = 0$. This way one has $l_2 \equiv 0$ in degree > 1. Now consider the map l_3. Define

$$l_3 = s \circ l_2l_2,$$

in degree 0 and then inductively by

$$l_3 = s \circ (l_2l_2 + l_3l_1),$$

in degree > 0. One checks that in degree 0

$$-l_1l_3 = -l_1s(l_2l_2) = l_2l_2 - (\lambda \circ \eta)(l_2l_2) = l_2l_2,$$

where $(\lambda \circ \eta)(l_2l_2) = 0$ since l_2l_2 is a boundary. While in degree > 0 we have

$$-l_1l_3 = -l_1(s(l_2l_2 + l_3l_1) = l_2l_2 + l_3l_1 + s\{l_1(l_2l_2 + l_3l_1)\} = l_2l_2 + l_3l_1,$$

where $l_1(l_2l_2 + l_3l_1) = 0$ since $l_2l_2 + l_3l_1$ is a boundary. So we have a well defined chain map l_3 satisfying $l_1l_3 + l_3l_1 + l_2l_2 = 0$.

Now consider l_3 on $X_1 \otimes X_0 \otimes X_0$: take an element $x_1 \otimes x_0 \otimes x_0' \in X_1 \otimes X_0 \otimes X_0$. By definition we have:
$(l_2l_2 + l_3l_1)(x_1 \otimes x_0 \otimes x_0') =$
$-sl_2l_1(l_2(x_1, x_0), x_0') + sl_2l_1(l_2(l_2(x_1, x_0'), x_0) - sl_2l_1(l_2(l_2(x_0, x_0'), x_1)$
$+s\{l_2(l_2(l_1x_1, x_0), x_0') - l_2(l_2(l_1x_1, x_0'), x_0) + l_2(l_2((x_0, x_0'), l_1x_1)\} =$

$-sl_2(l_2(l_1x_1, x_0), x_0') + sl_2(l_2(l_2(l_1x_1, x_0'), x_0) - sl_2(l_2(l_2(x_0, x_0'), l_1x_1)$
$+sl_2(l_2(l_1x_1, x_0), x_0') - sl_2(l_2(l_1x_1, x_0'), x_0) + sl_2(l_2((x_0, x_0'), l_1x_1) = 0.$

So $l_3 = s\{(l_2l_2 + l_3l_1)\} = s(0) = 0$ on $X_1 \otimes X_0 \otimes X_0$.

One can then proceed by induction to find that $l_3 = 0$ in degree > 1, for example take $x_1 \otimes x_1' \otimes x_0$ in $X_1 \otimes X_1 \otimes X_0$. $l_3(x_1 \otimes x_1' \otimes x_0) = s(l_2l_2 + l_3l_1)(x_1 \otimes x_1' \otimes x_0) = s\{l_2(l_2(x_1, x_1'), x_0) - l_2(l_2(x_1, x_0), x_1') + l_2(l_2(x_1', x_0), x_1) + l_3(l_1x_1 \otimes x_1' \otimes x_0) + l_3(x_1 \otimes l_1x_1' \otimes x_0)\} = 0$, since l_3 and l_2 are *zero* in degrees 1 and 2 respectively. Now consider the map l_4. Define

$$l_4 = s \circ (l_3l_2 - l_2l_3),$$

in degree 0, and then inductively by

$$l_4 = s \circ (l_3l_2 - l_2l_3 - l_4l_1),$$

in degree > 0. As before one can easily check that l_4 is a well-defined map that satisfies the corresponding sh-Lie relation at this step of the construction (i.e. $l_1l_4 - l_4l_1 + l_3l_2 - l_2l_3 = 0$).

Consider the value of l_4 on $x_0 \otimes x_0' \otimes x_0'' \otimes x_0''' \in X_0 \otimes X_0 \otimes X_0 \otimes X_0$. We find it convenient in the following calculation to use the identity $l_3(l_2(y_0, y_0'), y_0'', y_0''') = (sl_2l_2)(l_2(y_0, y_0'), y_0'', y_0''') = s\{l_2(l_2(l_2(y_0, y_0'), y_0''), y_0''') - l_2(l_2(l_2(y_0, y_0'), y_0'''), y_0'') + l_2(l_2(y_0'', y_0'''), l_2(y_0, y_0'))\}$, for $y_0, y_0', y_0'', y_0''' \in X_0$. By definition l_4 is the values of s on:
$l_3(l_2(x_0, x_0'), x_0'', x_0''') - l_3(l_2(x_0, x_0''), x_0', x_0''') + l_3(l_2(x_0, x_0'''), x_0', x_0'') +$
$l_3(l_2(x_0', x_0''), x_0, x_0''') - l_3(l_2(x_0', x_0'''), x_0, x_0'') + l_3(l_2(x_0'', x_0'''), x_0, x_0') -$
$l_2(l_3(x_0, x_0', x_0''), x_0''') + l_2(l_3(x_0, x_0', x_0'''), x_0'') - l_2(l_3(x_0, x_0'', x_0'''), x_0') +$
$l_2(l_3(x_0', x_0'', x_0'''), x_0) =$

$s\{l_2(l_2(l_2(x_0, x_0'), x_0''), x_0''') - l_2(l_2(l_2(x_0, x_0'), x_0'''), x_0'') + l_2(l_2(x_0'', x_0'''), l_2(x_0, x_0'))\} -$
$s\{l_2(l_2(l_2(x_0, x_0''), x_0'), x_0''') - l_2(l_2(l_2(x_0, x_0''), x_0'''), x_0') + l_2(l_2(x_0', x_0'''), l_2(x_0, x_0''))\} +$

$s\{l_2(l_2(l_2(x_0,x_0'''),x_0'),x_0'')-l_2(l_2(l_2(x_0,x_0''),x_0'''),x_0')+l_2(l_2(x_0',x_0''),l_2(x_0,x_0'''))\}+$
$s\{l_2(l_2(l_2(x_0',x_0''),x_0),x_0''')-l_2(l_2(l_2(x_0',x_0''),x_0'''),x_0)+l_2(l_2(x_0,x_0'''),l_2(x_0',x_0''))\}-$
$s\{l_2(l_2(l_2(x_0',x_0'''),x_0),x_0'')-l_2(l_2(l_2(x_0',x_0'''),x_0''),x_0)+l_2(l_2(x_0,x_0''),l_2(x_0',x_0'''))\}+$
$s\{l_2(l_2(l_2(x_0'',x_0'''),x_0),x_0')-l_2(l_2(l_2(x_0'',x_0'''),x_0'),x_0)+l_2(l_2(x_0,x_0'),l_2(x_0'',x_0'''))\}+$
$s\{l_2(l_1l_3(x_0,x_0',x_0''),x_0''')-l_2(l_1l_3(x_0,x_0',x_0'''),x_0'')+l_2(l_1l_3(x_0,x_0'',x_0'''),x_0')-$
$l_2(l_1l_3(x_0',x_0'',x_0'''),x_0)\}=$

$sl_2\{(l_2l_2+l_1l_3)(x_0,x_0',x_0''),x_0'''\}-sl_2\{(l_2l_2+l_1l_3)(x_0,x_0',x_0'''),x_0''\}+$
$sl_2\{(l_2l_2+l_1l_3)(x_0,x_0'',x_0'''),x_0'\}-sl_2\{(l_2l_2+l_1l_3)(x_0',x_0'',x_0'''),x_0\}+$
$l_2(l_2(x_0'',x_0'''),l_2(x_0,x_0'))+l_2(l_2(x_0,x_0'),l_2(x_0'',x_0'''))-l_2(l_2(x_0',x_0'''),l_2(x_0,x_0''))-$
$l_2(l_2(x_0,x_0''),l_2(x_0',x_0'''))+l_2(l_2(x_0',x_0''),l_2(x_0,x_0'''))+l_2(l_2(x_0,x_0'''),l_2(x_0',x_0''))=$
0 since $l_2l_2+l_1l_3=0$ in degree 0 and l_2 is skew-symmetric. So we have $l_4=0$ in degree 0.

Further l_4 is inductively found to be *zero* in higher degrees since $l_2=0$ and $l_3=0$ in degrees >1 and >0 respectively.

Next we inductively define for $n>4$,

$$l_n = s\circ(\sum_{i,j>1}(-1)^{i(j-1)}l_jl_i),$$

in degree 0, and

$$l_n = s\circ(\sum_{i+j=n+1,j>1}(-1)^{i(j-1)}l_jl_i),$$

in degree >0. (Again these are well defined maps for the sh-Lie structure.) The combination of maps (the l_k's within the s) in degree 0, and then inductively in degree >0, leads to 0 so that one has $l_n\equiv 0$ for $n>4$ (Notice that for l_5 in degree 0 one encounters l_3l_3, the inside l_3 raises the degree from 0 to 1 so that the combination is 0).

Summarizing:

Theorem 2.2 *Given a graded space X_* and a skew-symmetric linear map $\tilde{l}_2: X_0\bigotimes X_0\to X_0$ that satisfies conditions (i) and (ii), there exists an sh-Lie structure on X_* such that*
1. $l_2=0$ in degree >1.
2. $l_3=0$ in degree >0.
3. $l_n\equiv 0, n>3$.

Corollary 2.3 *Under the same hypotheses in the theorem there exists an sh-Lie structure on the graded space*

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X_2 \xrightarrow{i} X_1 \xrightarrow{l_1} X_0,$$

where X_1, X_0, $l_1 : X_1 \to X_0$, and $l_k, k > 1$ are as above, but with $X_2 = kerl_1$ and the inclusion $i : X_2 \to X_1$.

3 Examples

We will primarily consider three examples. The first of which fits perfectly into our discussion. It first appeared in [6] as the authors determined an sh-Lie structure on a Courant algebroid in the sense of the example given below. For convenience we recall the definition of a Courant algebroid.

Definition A Courant algebroid is a vector bundle $E \to M$ equipped with a nondegenerate symmetric bilinear form $< \cdot, \cdot >$ on the bundle, a skew-symmetric bracket $[\cdot, \cdot]$ on $\Gamma(E)$, and a bundle map $\rho : E \to TM$ such that the following properties are satisfied:
1. For any $e_1, e_2, e_3 \in \Gamma(E), J(e_1, e_2, e_3) = \mathcal{D}T(e_1, e_2, e_3)$;
2. for any $e_1, e_2 \in \Gamma(E), \rho[e_1, e_2] = [\rho e_1, \rho e_2]$;
3. for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M), [e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - < e_1, e_2 > \mathcal{D}f$;
4. $\rho \circ \mathcal{D} = 0$;
5. for any $e, h_1, h_2 \in \Gamma(E), \rho(e) < h_1, h_2 >=< [e, h_1] + \mathcal{D} < e, h_1 >, h_2 > + < h_1, [e, h_2] + \mathcal{D} < e, h_2 >>$;
where

$$J(e_1, e_2, e_3) = [[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2],$$

and $T(e_1, e_2, e_3)$ is the function on the base space M defined by

$$T(e_1, e_2, e_3) = \frac{1}{3} < [e_1, e_2], e_3 > +c.p.$$

($c.p.$ here denotes the cyclic permutations of the e_i's) and $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined such that the following identity holds

$$< \mathcal{D}f, e >= \frac{1}{2}\rho(e)f.$$

For more details on Courant algebroids we refer the reader to [6] and the references therein.

Example Let E be a Courant algebroid over a manifold M, and consider the sequence

$$\cdots \to 0 \to \ker\mathcal{D} \xrightarrow{i} C^\infty(M) \xrightarrow{\mathcal{D}} \Gamma(E)$$

where $i : \ker\mathcal{D} \to C^\infty(M)$ is the inclusion, and one assumes that $X_0 = \Gamma(E), X_1 = C^\infty(M)$, and $X_2 = \ker\mathcal{D}$. Define $\tilde{l}_2(e_1, e_2) = [e_1, e_2]$ (this is just l_2 in degree 0). It was shown in [6] that $\tilde{l}_2$ satisfies condition (i) as in Theorem 1.1, whereas condition (ii) from the same Theorem follows from the the first axiom in the above definition yielding an sh-Lie structure. The authors [6] have also shown that the sh-Lie structure has the explicit formulas

$$\begin{array}{lcll}
l_2(e_1, e_2) & = & [e_1, e_2] & \text{in degree } 0 \\
l_2(e, f) & = & < e, \mathcal{D}f > & \text{in degree } 1 \\
l_2 & = & 0 & \text{in degree} > 1 \\
l_3(e_1, e_2, e_3) & = & -T(e_1, e_2, e_3) & \text{in degree } 0 \\
l_3 & = & 0 & \text{in degree} > 0 \\
l_n & = & 0 & \text{for } n > 3.
\end{array}$$

What is interesting about this example is that the only nonzero maps are the same ones as in Theorem 2.2 and Corollary 2.3. In addition notice that the structure of this complex is similar to the "simplified" complex that appears in Corollary 2.3. ∎

Our next example comes from Lagrangian field theory, in particular it relates to the Poisson brackets of local functionals where the sh-Lie structure exists on a "DeRahm complex" as in [2]. We refer the reader to [2] and the references therein for more details regarding this subject.

Example let $E \to M$ be a vector bundle where the base space M is an n-dimensional manifold and let $J^\infty E$ be the infinite jet bundle of E. Consider the complex

$$\Omega^{0,0}(J^\infty E) \to \Omega^{1,0}(J^\infty E) \to \cdots \to \Omega^{n-1,0}(J^\infty E) \to \Omega^{n,0}(J^\infty E)$$

with a differential d_H which in local coordinates takes the form $d_H = dx^i D_i$, i.e. if $\alpha = \alpha_I dx^I$ then $d_H\alpha = D_i\alpha_I dx^i \wedge dx^I$. Here D_i is the total derivative derivation defined on the algebra of local functions on $J^\infty E$. It is defined by

$D_i = \dfrac{\partial}{\partial x^i} + u^a_{iJ}\dfrac{\partial}{\partial u^a_J}$ (we assume the summation convention, i.e., the sum is over all a and multi-index J). In this case $\tilde{l}_2$ was defined in [2] by

$$\tilde{l}_2(P\nu, Q\nu) = \omega(\mathbf{E}(P), \mathbf{E}(Q))\nu = \omega^{ab}(\mathbf{E}_b(Q))\mathbf{E}_a(P)\nu \qquad (3.2)$$

where $P\nu, Q\nu \in \Omega^{n,0}(J^\infty E)$ and $\mathbf{E}$ is the Euler-Lagrange operator with components

$$\mathbf{E}_a(P) = (-D)_I(\frac{\partial P}{\partial u^a_I}).$$

The bilinear mapping ω is a skew-symmetric total differential operator with the ω^{ab}'s as its components (see [2] for more details). It was shown in [2] that $\tilde{l}_2$ satisfies conditions (i) and (ii) as in Theorem 1.1. In fact $\tilde{l}_2$ satisfies condition (i) in a strong sense (with 0 on the right-hand side of the equation). Markl noted in [2] that with this strong condition the higher order maps can be chosen to be zero (The result in this paper's Theorem 2.2 is yet stronger since it does not require that the right-hand side of (i) be zero, only that it be a boundary). Here is a summary of the structure

$$\begin{array}{llcl}
l_2(P\nu, Q\nu) & = & \omega(\mathbf{E}(Q), \mathbf{E}(P))\nu & \text{in degree } 0 \\
l_2 & = & 0 & \text{in degree} > 0 \\
l_3(P\nu, Q\nu, R\nu) & \mathit{is} & \mathit{nonzero} & \text{in degree } 0 \\
l_3 & = & 0 & \text{in degree} > 0 \\
l_n & = & 0 & \text{for } n > 3. \quad \blacksquare
\end{array}$$

Our last example is within the context of symplectic manifolds, where we refer the reader to [1]. We include details in this example on how the sh-Lie structure maps are obtained. Notice that in this example the strong version of (i) of Theorem 1.1 does not hold in general, but our weaker hypothesis does hold.

Example Consider the following sequence

$$0 \to R \to \Omega^0(M) \to \Omega^1_C(M) \to 0,$$

where $\Omega^0(M)$ is the set of smooth real-valued functions on the symplectic manifold (M, ω) and $\Omega^1_C(M)$ is the set of closed one forms on M. We take $X_0 = \Omega^1_C(M), X_1 = \Omega^0(M)$ and $X_2 = R$. The chain map is $l_1 = i : R \to \Omega^0(M)$, and $l_1 = d : \Omega^0(M) \to \Omega^1_C(M)$, where i is the inclusion and d is the

differential operator. We then define a bilinear skew-symmetric map $\tilde{l}_2$ on $X_0 \times X_0$ by

$$\tilde{l}_2(\alpha, \beta) = \{\alpha, \beta\},$$

where $\{.,.\}$ is a Poisson bracket on $\Omega^1(M)$ (e.g. see definition 3.3.7 in [1]). Notice that $\tilde{l}_2$ satisfies the two conditions needed to guarantee the existence of an sh-Lie algebra (Theorem 1.1).

Now to extend $\tilde{l}_2$, first take an element in $X_1 \otimes X_0$ say $f \otimes \beta$, then $\tilde{l}_2 l_1(f \otimes \beta) = \tilde{l}_2(df \otimes \beta) = \{df, \beta\}$. Now notice that $h = L_{\beta\#} f + c = -i_{X_f}\beta + L_{\beta\#} f + i_{X_f} i_{\beta\#}\omega + c \in X_1$ satisfies

$$\begin{aligned} l_1(h) &= d(-i_{X_f}\beta + L_{\beta\#} f + i_{X_f} i_{\beta\#}\omega + c) \\ &= -d i_{X_f}\beta - i_{X_f} d\beta + d L_{\beta\#} f + d(i_{X_f} i_{\beta\#}\omega) \\ &= -L_{X_f}\beta + L_{\beta\#} df + d(i_{X_f} i_{\beta\#}\omega) \\ &= \{df, \beta\}, \end{aligned}$$

where ω is the *symplectic* 2-form on M. So if we take $c = 0$ we get $l_2(f \otimes \beta) = L_{\beta\#} f$. Then l_2 would be defined on $X_0 \otimes X_1$ by skew-symmetry. To proceed take an element in $X_1 \otimes X_1$ say $f \otimes g$, and notice that $l_2 l_1(f \otimes g) = l_2(df \otimes g - f \otimes dg) = L_{df\#} g - (-L_{dg\#} f) = L_{X_f} g + L_{X_g} f = 0$. Hence l_2 on $X_1 \otimes X_1$ is *zero*. Now take an element in $X_2 \otimes X_0$ say $k \otimes \beta$, then $l_2 l_1(k \otimes \beta) = l_2(k \otimes \beta + 0) = L_{\beta\#} k = 0$, since k is a constant function. Therefore l_2 is *zero* on $X_2 \otimes X_0$. By skew-symmetry l_2 will also be *zero* on $X_0 \otimes X_2$.

Next we turn to l_3, take an element in $X_0 \otimes X_0 \otimes X_0$, say $\alpha \otimes \beta \otimes \gamma$, then $l_2 l_2$ maps it into $\{\{\alpha, \beta\}, \gamma\} - \{\{\alpha, \gamma\}, \beta\} + \{\{\beta, \gamma\}, \alpha\})$ which is 0 (The Jacobi identity). So we have $l_3 = 0$ on $X_0 \otimes X_0 \otimes X_0$. Now take $f \otimes \beta \otimes \gamma \in X_1 \otimes X_0 \otimes X_0$. Under $l_2 l_2 + l_3 l_1$ it is mapped to

$$\begin{gathered} l_2(L_{\beta\#} f, \gamma) - l_2(L_{\gamma\#} f, \beta) + l_2(\{\beta, \gamma\}, f) + 0 = \\ L_{\gamma\#} L_{\beta\#} f - L_{\beta\#} L_{\gamma\#} f - L_{\{\beta,\gamma\}\#} f = \\ L_{\{\beta,\gamma\}\#} f - L_{\{\beta,\gamma\}\#} f = 0. \end{gathered}$$

So l_3 is *zero* on $X_1 \otimes X_0 \otimes X_0$. Utilizing skew-symmetry we have $l_3 = 0$ in degree 1. On higher degrees l_3 is trivially *zero* since $X_n = 0$ for $n > 2$, and furthermore $l_n = 0$ for $n > 3$. To summarize

$$\begin{array}{lcll} l_2((\alpha, \beta) & = & \{\alpha, \beta\} & \text{in degree } 0 \\ l_2(f, \beta) & = & L_{\beta\#} f & \text{in degree } 1 \\ l_2 & = & 0 & \text{in degree} > 1 \\ l_n & = & 0 & \text{for } n > 2. \end{array}$$

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